29 September 2022

Euler formula $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

Proof using power series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$ sub $x = i\theta$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$ $x = \theta$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + - \cdots$$ $x = \theta$

Prove it by the suggested substitutions

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!}$$

$$\cos\theta + i\sin\theta = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!}$$

Proof by differentiation $\quad e^{i\theta} = \cos\theta + i\sin\theta$

consider $f(\theta) = \dfrac{\cos\theta + i\sin\theta}{e^{i\theta}}$ *

$f(\theta) = e^{-i\theta}(\cos\theta + i\sin\theta)$

find $f'(\theta)$ using the product rule

$u = e^{-i\theta} \qquad v = \cos\theta + i\sin\theta$

$u' = -ie^{i\theta} \qquad v' = -\sin\theta + i\cos\theta$

$$f'(\theta) = e^{-i\theta}(i\cos\theta - \sin\theta) - ie^{-i\theta}(\cos\theta + i\sin\theta)$$
$$= e^{-i\theta}(i\cos\theta - \sin\theta) + e^{-i\theta}(-i\cos\theta + \sin\theta)$$
$$= 0$$

$f'(\theta)$ is 0 for all θ

$f(\theta)$ is a horizontal line

$f(0) = 1 \qquad f(\theta) = 1$ into original formula *

$$1 = \frac{\cos\theta + i\sin\theta}{e^{i\theta}} \Rightarrow e^{i\theta} \equiv \cos\theta + i\sin\theta$$

$z = a + ib = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$

cartesian form   polar form   exponential form

Q1 convert to exponential form

$z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $w = 1 + i\sqrt{3}$

need to familiar with the unit circle

Solution

$|z| = 1$ $\arg(z) = \frac{\pi}{3}$

$z = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = e^{i\pi/3}$

$|w| = 2$ $\arg(w) = \frac{\pi}{3}$

$w = 2(\cos\theta + i\sin\theta) = 2e^{i\pi/3}$

Q2 $z = e^{i\pi/3}$ and $w = 2e^{i\pi/3}$

plot the follow on the Argand diagram

z, z^2, z^3 and $w, w^2,$ and w^3

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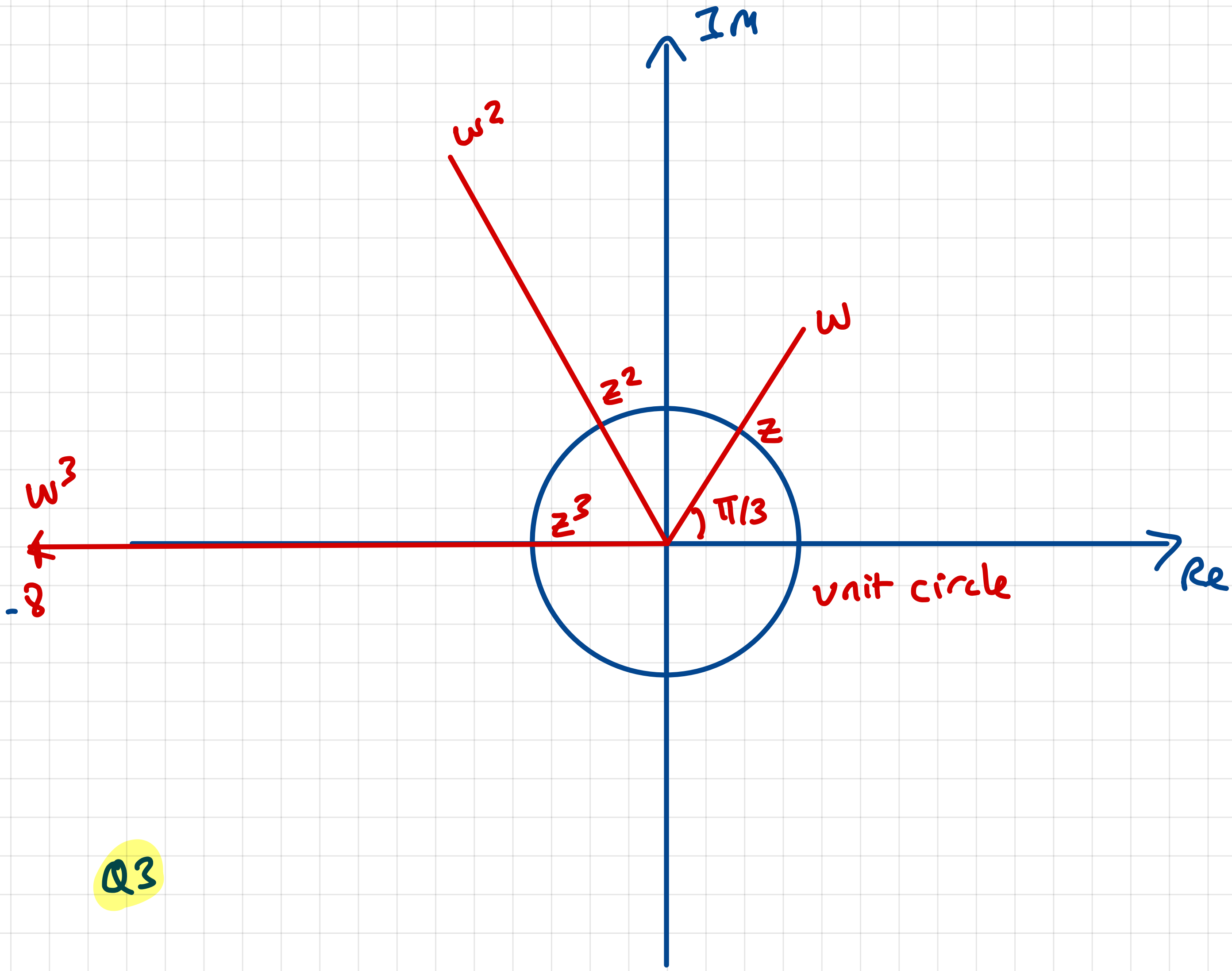


Q3

$z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ find z^3 almost instantly

$z^3 = -1$

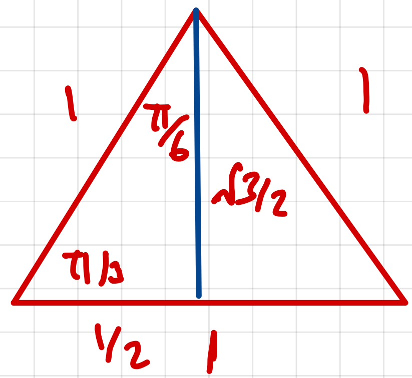
$w = 1 + i\sqrt{3}$ find w^3 almost instantly

$w^3 = -8$

$z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \qquad w = \frac{\sqrt{3}}{2} + \frac{i}{2}$

(i) find $\frac{z}{w}$ in cartezian form

(ii) write z and w in exponential form hence find $\arg\left(\frac{z}{w}\right)$



(iii) deduce a surd form for :

$\cos\frac{\pi}{12}$ and $\sin\frac{\pi}{12}$

Solution

(i) $$\frac{z}{w} = \frac{\frac{1}{\sqrt{2}}(1+i)}{\frac{1}{2}(\sqrt{3}+i)} = \frac{2}{\sqrt{2}} \frac{(1+i)}{(\sqrt{3}+i)}$$

$$\frac{2}{\sqrt{2}} \frac{(1+i)}{(\sqrt{3}+i)} \frac{(\sqrt{3}-i)}{(\sqrt{3}-i)} = \frac{\cancel{2}\,1}{\sqrt{2}} \frac{\sqrt{3}+1 \; i(\sqrt{3}-1)}{\cancel{4}\,2}$$

(ii) $|z| = 1 \qquad |w| = 1$

$\text{Arg}(z) = \pi/4 \qquad \text{Arg}(w) = \frac{\pi}{6}$

$z = e^{i\pi/4}$

$$\frac{z}{w} = \frac{e^{i\pi/4}}{e^{i\pi/6}} = e^{i\pi/4} e^{-\pi/6} = e^{i\pi/12}$$

$$\frac{z}{w} = e^{i\pi/12} = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12} = \frac{1}{\sqrt{2}}\left(\sqrt{3}+1 + i(\sqrt{3}-1)\right)$$

(iii) $\Rightarrow \quad \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{\sqrt{2}} \qquad \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{\sqrt{2}}$

Another way to find this

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\theta = \frac{\pi}{12}$$

$$\cos\frac{\pi}{6} = 2\cos^2\frac{\pi}{12} - 1$$

$$\frac{\sqrt{3}}{2} = 2\cos^2\frac{\pi}{12} - 1$$

$$\cos\theta + i\sin\theta = e^{i\theta}$$

$$\cos(\text{something}) + i\sin(\text{something}) = e^{i\,\text{something}}$$

square both sides

$$(\cos\theta + i\sin\theta)^2 = (e^{i\theta})^2$$

$$(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta) = e^{i2\theta}$$

$$\cos^2\theta + 2i\sin\theta\cos\theta + i^2\sin^2\theta \equiv e^{i2\theta}$$

$$\cos^2\theta - \sin^2\theta + i\,2\sin\theta\cos\theta \equiv e^{i2\theta}$$

consider $e^{i2\theta} = \cos 2\theta + i\sin 2\theta$

apply the something here

$$\Rightarrow \left.\begin{aligned} \cos^2\theta - \sin^2\theta &= \cos 2\theta \\ 2\sin\theta\cos\theta &= \sin 2\theta \end{aligned}\right\} \text{double angle formulas}$$

next lesson

find the "triple angle" formulas for $\cos 3\theta$ and $\sin 3\theta$ using the above method