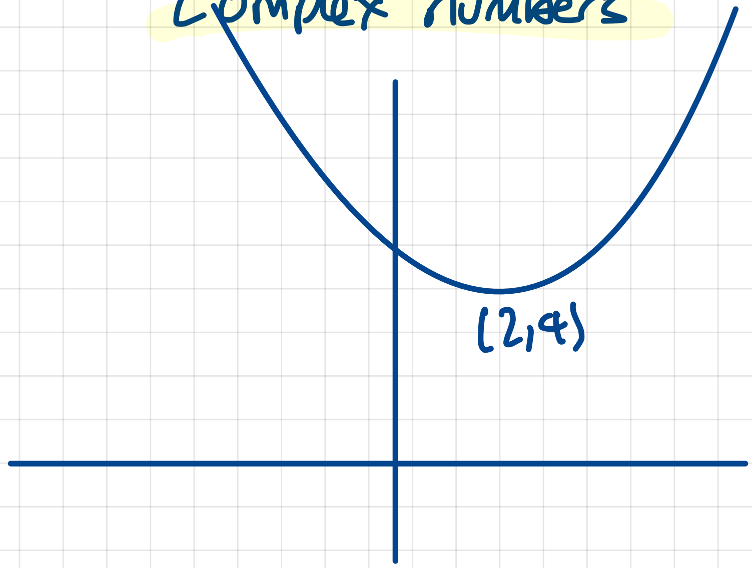
26 September 2022

Complex numbers



started

$y = x^2$

$y = (x-2)^2 + 4$

solutions which are not "real" set $y = 0$

$0 = (x-2)^2 + 4$

$-4 = (x-2)^2$

square root both sides

$\pm\sqrt{-4} = x - 2$

$\sqrt{-4} = \sqrt{-1 \times 4}$

$= \sqrt{-1}\sqrt{4}$

used $\sqrt{ab} = \sqrt{a}\sqrt{b}$

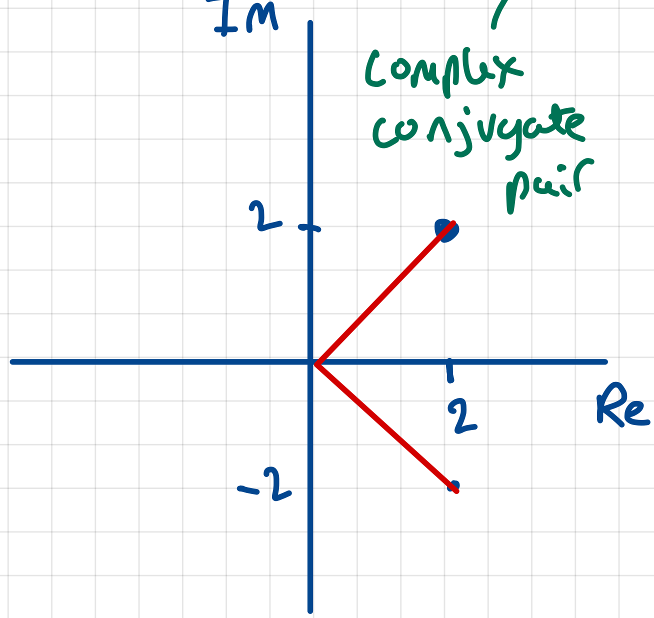
surd law $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

$\sqrt{-1} = i \Rightarrow i^2 = -1$

$\pm 2i = x - 2$

solutions are $x = 2 \pm 2i$

complex conjugate pair



Argand Diagram

$x = \underbrace{2 + 2i}_{\text{complex number}}$

imaginary part

real part

Q1 consider $f(x) = x^2 - 6x + 10$

find the complex solutions in the form

$z = a \pm ib$

show that $f(a+ib) = 0$

and $f(a-ib) = 0$

Solution

$0 = x^2 - 6x + 9 - 9 + 10$

$0 = (x-3)^2 + 1$

$-1 = (x-3)^2$

$\pm i = x - 3$

$x = 3 \pm i$

$f(3+i) = (3+i)^2 - 6(3+i) + 10$

$= (3+i)(3+i) - 18 - 6i + 10$

$= 9 + \cancel{6i} + i^2 - 8 - \cancel{6i}$

$= 9 - 1 - 8 = 0$

$f(3-i) =$ finish off yourself

$z = a + ib$
$z^* = a - ib$ } complex conjugate pair

sometimes $\bar{z}$

$zz^* = (a+ib)(a-ib) = a^2 + b^2$

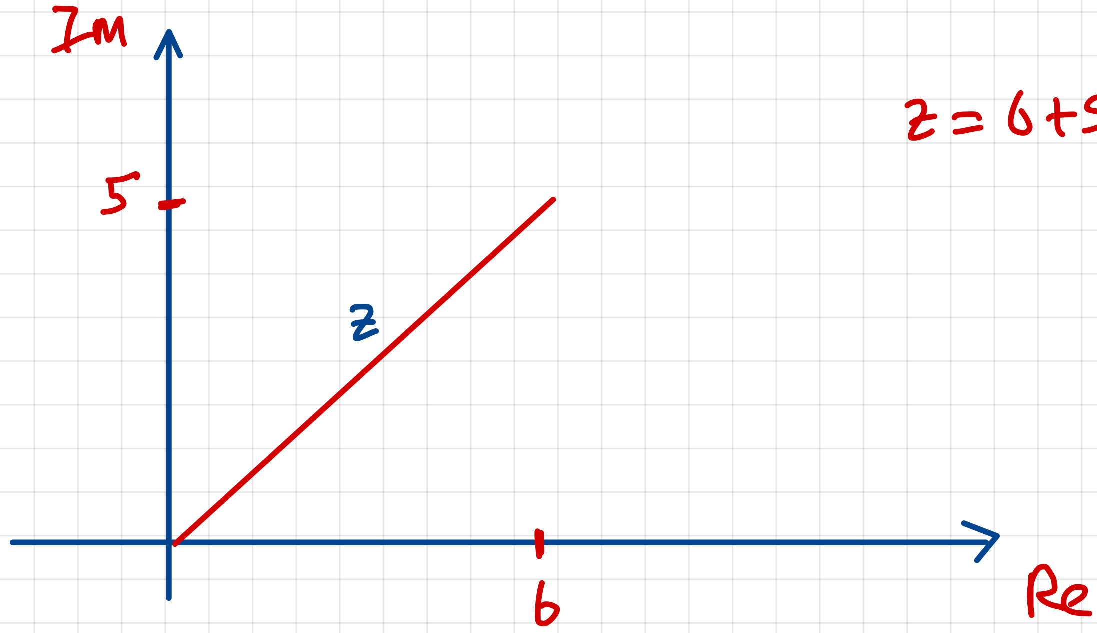
A complex number multiplied by it's conjugate is a real quantity

Try $z = 3 + 2i$

$zz^* = (3+2i)(3-2i)$

$= 9 - 6i + 6i - 4i^2$

$= 13$



$z = 6 + 5i$

$|z|$ modulus of z, this is the physical length of the red line segment as seen on the paper

$$|z| = \sqrt{6^2 + 5^2} = \sqrt{61}$$

$z = a + ib$ $\qquad$ $|z| = \sqrt{a^2 + b^2}$

$z z^* = a^2 + b^2$ $\qquad$ $z z^* = |z|^2$

Previously

Addition, subtraction, multiplication, analogous to real numbers, division is different

$$\frac{2+3i}{1+i}$$

multiply by conjugate of denominator

$$\frac{(2+3i)(1-i)}{(1+i)(1-i)} = \frac{2 - 2i + 3i - 3i^2}{2}$$

$$= \frac{5+i}{2} = \frac{5}{2} + \frac{i}{2}$$

Q2 write $\frac{1+i}{2-i}$ in the form $a+ib$

$$\frac{(1+i)(2+i)}{(2-i)(2+i)} = \frac{2+i+2i+i^2}{5} = \frac{1+3i}{5} = \frac{1}{5} + \frac{3i}{5}$$

Q3 $\frac{1+i}{2+ib}$ is a real number hence find the value of b.

Method 1

$$\frac{(1+i)(2-ib)}{(2+ib)(2-ib)} = \frac{2-ib+2i-i^2b}{2^2+b^2}$$

$$= \frac{2+b+i(2-b)}{2^2+b^2}$$

no imaginary part

hence $2-b=0$

$b=2$

Method 2

$$\frac{1+i}{2+ib} = \frac{1+i}{2(1+ib/2)}$$

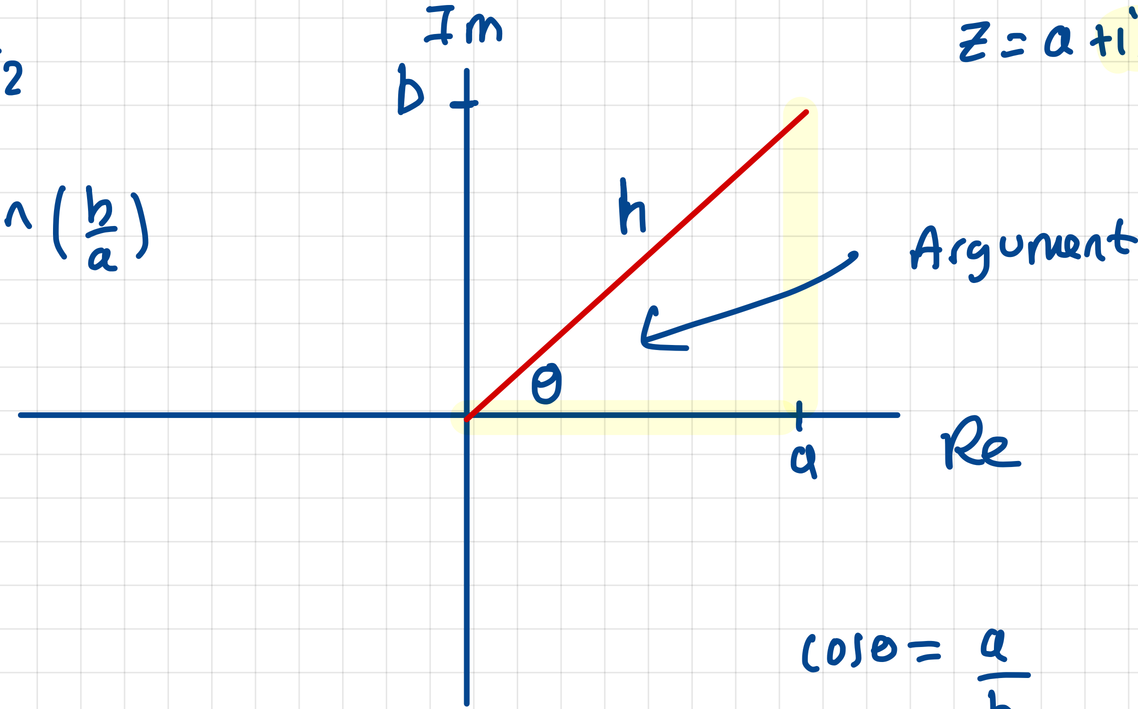
if $b/2=1$ simplified to $\frac{1}{2}$ $b=2$

Polar form for a complex number

$0 < \theta < \pi/2$

$z = a + ib$

$\theta = \arctan\left(\frac{b}{a}\right)$



$\cos\theta = \frac{a}{h}$

$|z| = \sqrt{a^2 + b^2}$

$a = h\cos\theta$

$a = |z|\cos\theta$

$b = h\sin\theta$

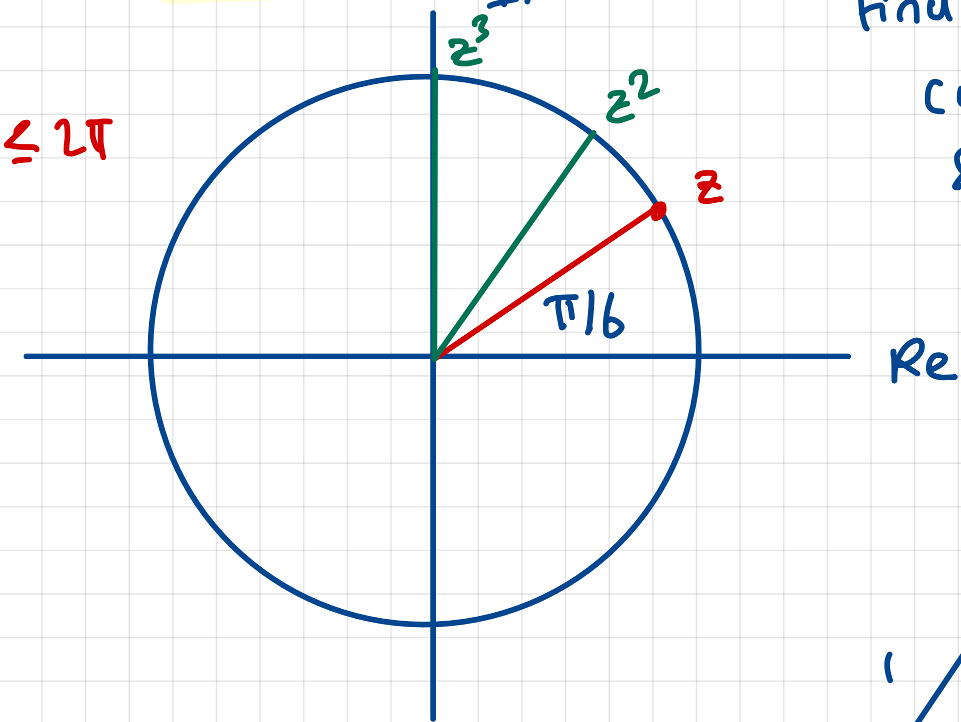
$b = |z|\sin\theta$

$z = |z|\cos\theta + i\,|z|\sin\theta$

$z = |z|(\cos\theta + i\sin\theta)$

Q1

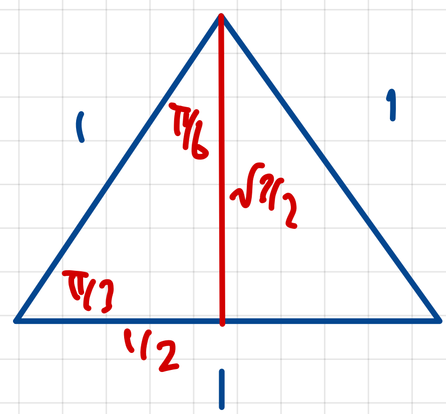
Unit circle



$0 \le \theta \le 2\pi$

Find z in cartezian form & polar form

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ $\sin\frac{\pi}{6} = \frac{1}{2}$



$z = \frac{\sqrt{3}}{2} + \frac{i}{2}$ polar form $z = 1\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

modulus   argument

(i) z^2 and $|z^2|$ and the argument of z^2

(ii) find z^3 and $|z^3|$ and the argument of z^3

Solution

$$z^2 = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$$

$$= \frac{3}{4} + \frac{\sqrt{3}}{4}i + \frac{\sqrt{3}}{4}i + \frac{i^2}{4}$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$|z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

Argument $= \pi/3$

$\cos\theta = 1/2$ $\sin\theta = \sqrt{3}/2$

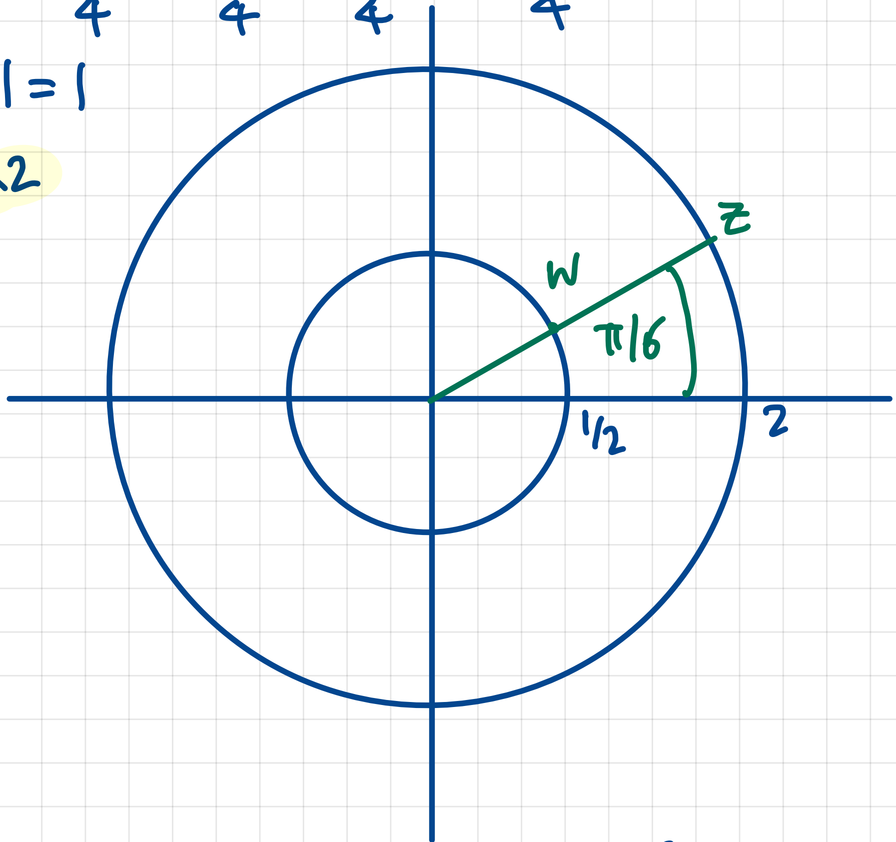
$z^2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$z^3 = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

$z^3 = \frac{\sqrt{3}}{4} + \frac{i}{4} + \frac{3i}{4} + \frac{\sqrt{3}}{4}i^2 = i$

$|z^3| = 1$

Q2



write z in cartezian form

write w in cartezian form

(i) z^2 and $|z^2|$ and the argument of z^2

(ii) find z^3 and $|z^3|$ and the argument of

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