3 October 2022

last time $\cos\theta + i\sin\theta = e^{i\theta}$

powers to both sides

homework $(\cos\theta + i\sin\theta)^3 = (e^{i\theta})^3$

$(\cos\theta + i\sin\theta)^3 = e^{i3\theta} = \cos 3\theta + i\sin 3\theta$

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta = e^{in\theta} = (e^{i\theta})^n$

Expand using Pascal triangle

$\cos 3\theta =$ real part

$\sin 3\theta =$ imaginary part

Compound Angle formulas proof

Start $\cos\theta + i\sin\theta = e^{i\theta}$ (A) $\cos\phi + i\sin\phi = e^{i\phi}$ (B)

p..d.ct

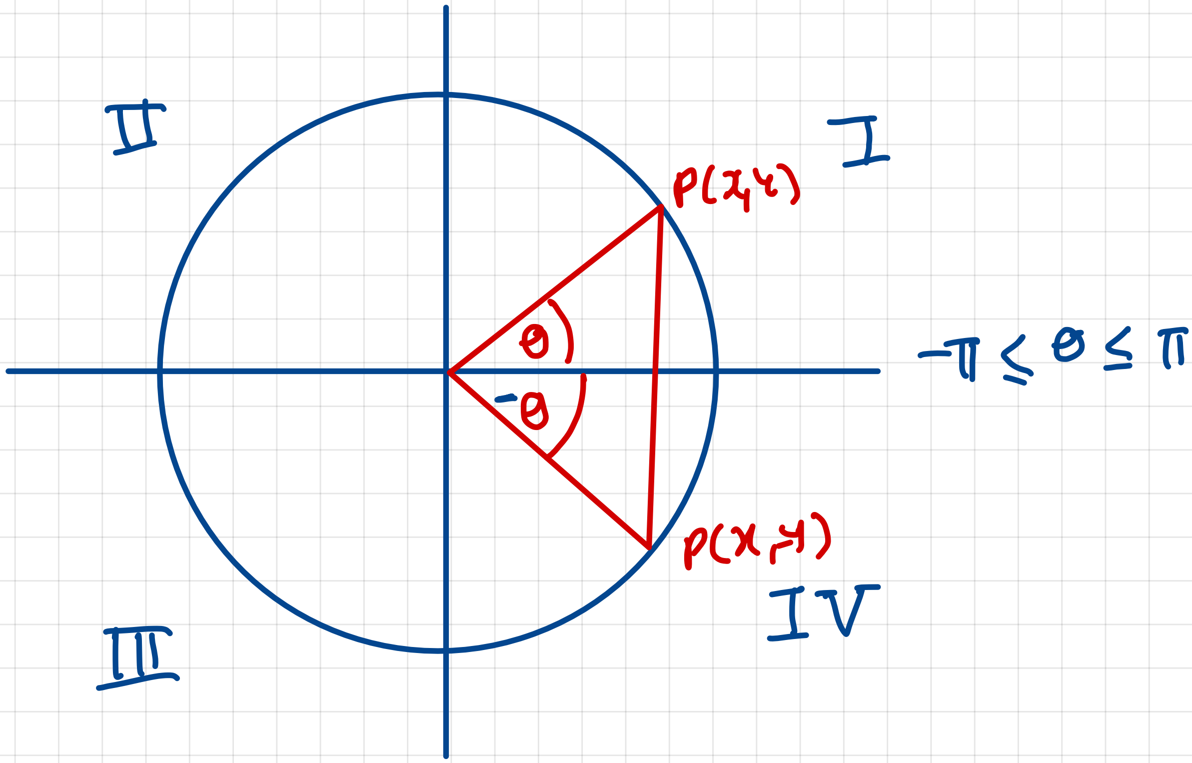
(A) x (B) $(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = e^{i\theta}e^{i\phi}$

$\cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + \cos\theta\sin\phi) = e^{i(\theta+\phi)}$

$= \cos(\theta+\phi) + i\sin(\theta+\phi)$

$$\left.\begin{aligned}\cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin(\theta+\phi) &= \sin\theta\cos\phi + \cos\theta\sin\phi\end{aligned}\right\} \text{ if } \theta=\phi$$

Do you have the unit circle at your finger tips?



prove $\cos\theta = \cos(-\theta)$ using the unit circle

$\sin\theta = -\sin(-\theta)$

fundamental Definition of the unit circle

$x = \cos\theta$ $\qquad$ $y = \sin\theta$

$x = \cos(-\theta)$ $\qquad$ $-y = \sin(-\theta)$

$\Rightarrow \cos\theta = \cos(-\theta)$ $\qquad$ $\sin\theta = -\sin(-\theta)$

$$e^{-i\theta} = \frac{1}{e^{i\theta}} = \cos(-\theta) + i\sin(-\theta)$$

$$= \cos\theta - i\sin\theta$$

$e^{-i\theta} = \cos\theta - i\sin\theta$ from the unit circle

Start $\cos\theta + i\sin\theta = e^{i\theta}$ (A) $\cos\phi + i\sin\phi = e^{i\phi}$ (B)

d_v_de to prove the remaining compound angle identities

$$\frac{e^{i\theta}}{e^{i\phi}} = e^{i(\theta-\phi)} = e^{i\theta}e^{-i\theta} = (\cos\theta + i\sin\theta)(\cos(-\phi) + i\sin(-\phi))$$

$$= (\cos\theta + i\sin\theta)(\cos\phi - i\sin\phi)$$

expand

$$= \cos(\theta-\phi) + i\sin(\theta-\phi)$$

equate the real and imaginary parts with this

Applications to powers of $\cos^n\theta$ $\sin^n\theta$

$z = \cos\theta + i\sin\theta$ $\quad z^{-1} = \cos\theta - i\sin\theta$

$2\cos\theta = z + \frac{1}{z} \Rightarrow \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$2i\sin\theta = z - \frac{1}{z} \Rightarrow \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

$z^2 = \cos 2\theta + i\sin 2\theta$ $\quad z^{-2} = \cos 2\theta - i\sin 2\theta$

$2\cos 2\theta = z^2 + \frac{1}{z^2}$

$2i\sin 2\theta = z^2 - \frac{1}{z^2}$

$z^3 = \cos 3\theta + i\sin 3\theta$ $\quad z^{-3} = \cos 3\theta - i\sin 3\theta$

$2\cos 3\theta = z^3 + \frac{1}{z^3}$

$2i\sin 3\theta = z^3 - \frac{1}{z^3}$

$z^n = \cos(n\theta) + i\sin(n\theta)$ $\quad z^{-n} = \cos(n\theta) - i\sin(n\theta)$

$2\cos(n\theta) = z^n + \frac{1}{z^n}$ $\quad 2i\sin(n\theta) = z^n - \frac{1}{z^n}$

$z^n = \cos(n\theta) + i\sin(n\theta)$ $\quad z^{-n} = \cos(n\theta) - i\sin(n\theta)$

$2\cos(n\theta) = z^n + \frac{1}{z^n}$ $\quad 2i\sin(n\theta) = z^n - \frac{1}{z^n}$

$n=1$

$2\cos\theta = z + \frac{1}{z}$

square both sides

$2^2\cos^2\theta = \left(z+\frac{1}{z}\right)^2 = z^2 + 1 + 1 + \frac{1}{z^2}$

$4\cos^2\theta = z^2 + \frac{1}{z^2} + 2$

$4\cos^2\theta = 2\cos 2\theta + 2$

$2\cos^2\theta = \cos 2\theta + 1 \Rightarrow \cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

Q1 $\quad (2\cos\theta)^3 = \left(z+\frac{1}{z}\right)^3$ expand and simplify the R.H.S

hence find

$\int \cos^3\theta \, d\theta$

1
1 2 1
1 3 3 1

$8\cos^3\theta = z^3 + 3z^2\frac{1}{z} + 3z\frac{1}{z^2} + \frac{1}{z^3}$

$8\cos^3\theta = z^3 + \frac{1}{z^3} + 3\left(z+\frac{1}{z}\right) = 2\cos 3\theta + 6\cos\theta$

$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$

$\int \cos^3\theta \, d\theta = -\frac{1}{12}\sin 3\theta + \frac{3}{4}\sin\theta + c$