Geometric Series

26 August 2022

$8, 4, 2, 1, 1/2, 1/4, \ldots$ $a=8$ $r=1/2$

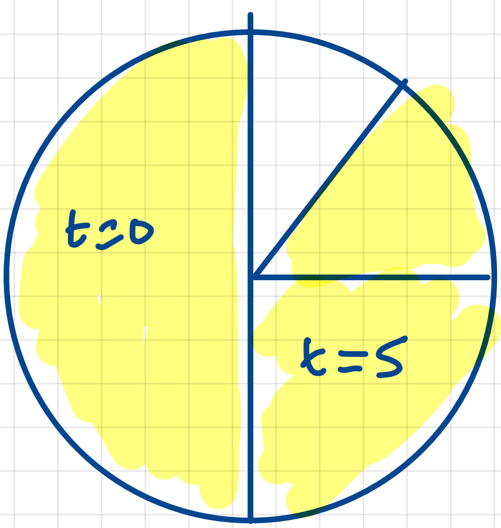
$2, 4, 8, 16, 32, 64, \ldots$ $a=2$ $r=2$

$0.1, 0.01, 0.001, 0.0001, \ldots$ $a=0.1$ $r=1/10$

$a, ar, ar^2, ar^3, \ldots$

a = the first term r = common ratio

Will the sum of an infinite series of positive numbers add up to ∞ (infinity)



cake

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

$$= 1$$

$$\sum_{1}^{\infty} \frac{1}{2n} = \infty$$

short hand for

$$\begin{array}{cccc} n=1 & n=2 & n=3 & n=4 \\ \frac{1}{2} + & \frac{1}{4} + & \frac{1}{8} + & \frac{1}{16} \end{array}$$

$$\sum_{1}^{4} n^2 = 1^2 + 2^2 + 3^2 + 4^2$$

$$\sum_{n=0}^{5} 2^n = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5$$

Sigma notation

Formula for the sum of a geometric series

$u_1 = a \quad u_2 = ar \quad u_3 = ar^2 \quad u_4 = ar^3 \quad u_5 = ar^4$

$u_n = ar^{n-1}$

$a, ar, ar^2, ar^3, ar^4, ar^5$

$a > 0$

$r = 1$ it's not a geometric series

$r > 1$ next term is bigger than the previous term

$r < 1$ next term is smaller than the previous terms

$n \to \infty \quad u_n \to 0$

$$S_4 = a + ar + ar^2 + ar^3 \quad (A)$$

$$rS_4 = ar + ar^2 + ar^3 + ar^4 \quad (B)$$

$(B) - (A)$

$$rS_4 - S_4 = ar^4 - a$$

$$S_4(r-1) = a(r^4-1)$$

divide both sides by $(r-1)$

$$S_4 = \frac{a(r^4-1)}{(r-1)}$$

can you derive a formula for S_n

$$S_n = a + ar + ar^2 + \ldots\ ar^{n-1}$$

$$rS_n = ar + ar^2 + ar^3 + \ldots\ ar^n$$

$$rS_n - S_n = ar^n - a$$

$$S_n(r-1) = a(r^n-1)$$

$$S_n = a\frac{(r^n-1)}{r-1}\frac{(-1)}{(-1)} = a\frac{(1-r^n)}{1-r}$$

$$3 \times (4-2) \times (-1)$$

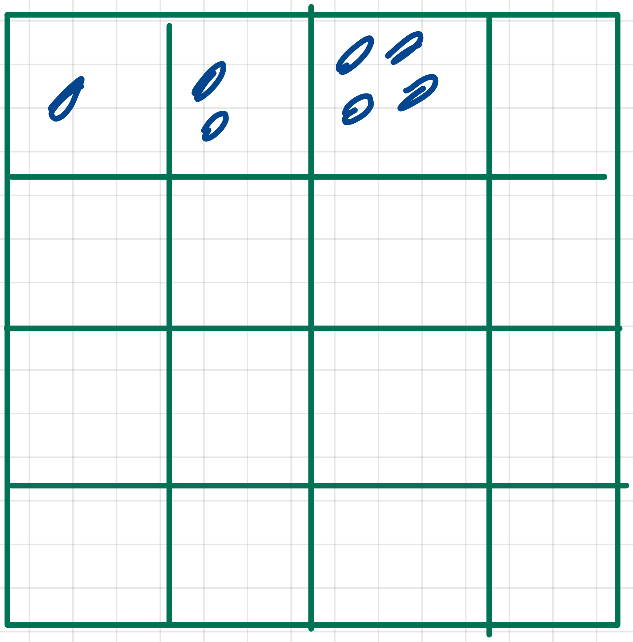
$$= -3 \times (4-2)$$

$$= 3(2-4)$$

$$\frac{3-2}{5-3} = \frac{(3-2)(-1)}{(5-3)(-1)} = \frac{2-3}{3-5} = \frac{-1}{-2} = \frac{1}{2}$$

$$4 \times \frac{2}{4} = \frac{4 \times 2}{4}$$

Q1



Reduced chess board

how many grains of rice on the board?

$a=1$ $r=2$ $n=16$

$$S_{16} = \frac{1(2^{16}-1)}{2-1} = 65\,535$$

8 by 8 chess board

$$S_{64} = \frac{1(2^{64}-1)}{2-1}$$

Q2 find the sum of the first 10 terms of the geometric series

$$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27} \ldots$$

$a = 1 \qquad r = 1/3$

$$S_{10} = 1\left(\left(\frac{1}{3}\right)^{10} - 1\right) \div \left(\frac{1}{3} - 1\right)$$

$$S_{10} = \left(\frac{1}{3^{10}} - 1\right) \div -\frac{2}{3}$$

$$S_{10} = \left(\frac{1}{3^{10}} - 1\right) - \frac{3}{2} = \frac{3}{2}\left(1 - \frac{1}{3^{10}}\right)$$

$$= \frac{3}{2} - \frac{3}{2 \times 3^{10}}$$

$$S_{10} = \frac{3}{2} - \frac{1}{2 \times 3^{9}}$$

Q2 b find S_{20}, S_{50}, for the series and also S_{1000}, don't use your calculator

Solution

$$S_{10} = \frac{3}{2} - \frac{1}{2\times 3^{9}} \qquad S_{50} = \frac{3}{2} - \frac{1}{2\times 3^{49}}$$

$$S_{20} = \frac{3}{2} - \frac{1}{2\times 3^{19}} \qquad S_{10} = \frac{3}{2} - \frac{1}{2\times 3^{99}}$$

$$\underset{n\to\infty}{S_n \to 3/2} \qquad \frac{\text{some number}}{\text{really really Big number}} \approx 0$$

$$\underset{n\to\infty}{\frac{1}{2^n} \to 0}$$

$$S_n = \frac{a(r^n - 1)}{r-1} \qquad |r| < 1$$

$$-1 < r < 1$$

$$r \neq 0$$

$$n \to \infty \quad r^n \to 0$$

e.g. $r = 1/3$ $\quad \left(\frac{1}{3}\right)^n = \frac{1^n}{3^n} = \frac{1}{3^n} \to 0$

$$n \to \infty$$

$$S_\infty = \frac{-a}{r-1} = \frac{a}{1-r}$$

Example find the sum to infinity

$$0.3 + 0.03 + 0.003 + 0.0003 + \ldots\ldots$$

$a = 0.3 \qquad r = 0.1$

$$S_\infty = \frac{0.3}{1-0.1} = \frac{0.3}{0.9} = \frac{1}{3}$$

Q2 find the fractional form of $0.\dot{2}\dot{7}$

$$0.27 + 0.0027 + 0.000027 + \ldots\ldots$$

$a = 0.27 \qquad r = 0.01$

$$S_\infty = \frac{0.27}{1-0.01} = \frac{0.27}{0.99} = \frac{3}{11}$$

$$\frac{0.27}{0.99} = \frac{27}{99} = \frac{3 \times \cancel{9}}{11 \times \cancel{9}} = \frac{2}{11}$$