Roots of unit

6 October 2022

$$z^2 = 1$$

By inspection it's 1 and -1

The values of z that satisfies the equation are the "root" same as the solution, often they are labelled as ω

$$z^2 - 1 = 0$$

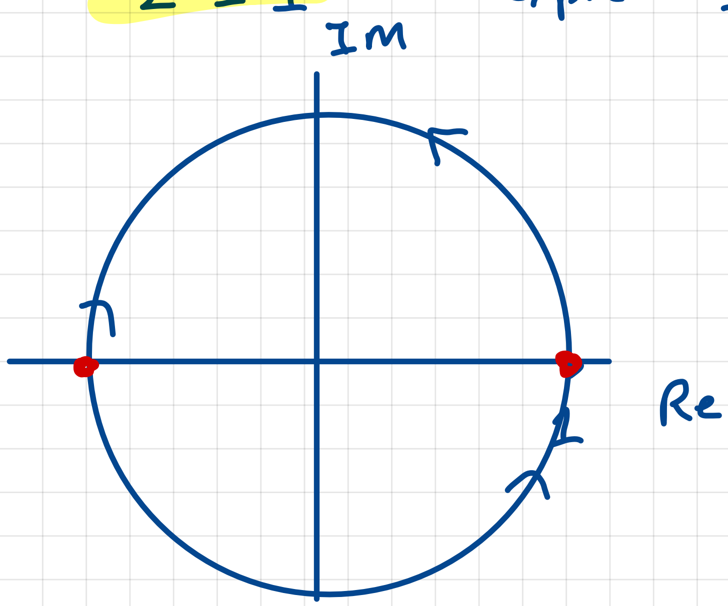
$(z+1)(z-1) = 0$ difference of two square

solutions $z = 1$ and $z = -1$

roots are $w_0 = 1$ $w_1 = -1$ agree that $w_1^2 = w_0$

Now it get zany

$z^2 = 1$ express 1 in exponential form



argument $0, 2\pi, 4\pi, 2n\pi$

$$z^2 = e^{i2n\pi}$$

$$(z^2)^{1/2} = (e^{i2n\pi})^{1/2}$$

$$z = e^{in\pi}$$

$n=0$ $z = e^0 = 1$

$n=1$ $z = e^{i\pi} = \cos\pi + i\sin\pi$

$$z = -1$$

$n=2$ $z = e^{i2\pi}$

$= \cos(2\pi) + i\sin(2\pi)$

$= 1$

increase n, repeat solution, we stop

$z^3 = 1$ $\qquad$ $z^3 - 1 = (z-1)(z^2 + z + 1)$

$$z^3 - 1 = (z-1)\left(z^2 + z + \left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 + 1\right)$$
$$= (z-1)\left(\left(z + \tfrac{1}{2}\right)^2 + \tfrac{3}{4}\right)$$

solution $z = 1$ $\qquad$ $\left(z + \frac{1}{2}\right)^2 = -\frac{3}{4}$

$$z + \frac{1}{2} = \pm i \frac{\sqrt{3}}{2}$$

$$z = i\frac{\sqrt{3}}{2} - \frac{1}{2} \qquad z = -i\frac{\sqrt{3}}{2} - \frac{1}{2}$$

Complex numbers way

$z^3 = 1$ $\qquad$ $z^3 = e^{i2n\pi}$

$$(z^3)^{1/3} = \left(e^{i2n\pi}\right)^{1/3}$$

$$z = e^{i2n\pi/3}$$

$n=0$ $\quad$ $w_0 = e^0 = 1$

$n=1$ $\quad$ $w_1 = e^{i2\pi/3}$

$n=2$ $\quad$ $w_2 = e^{i4\pi/3}$

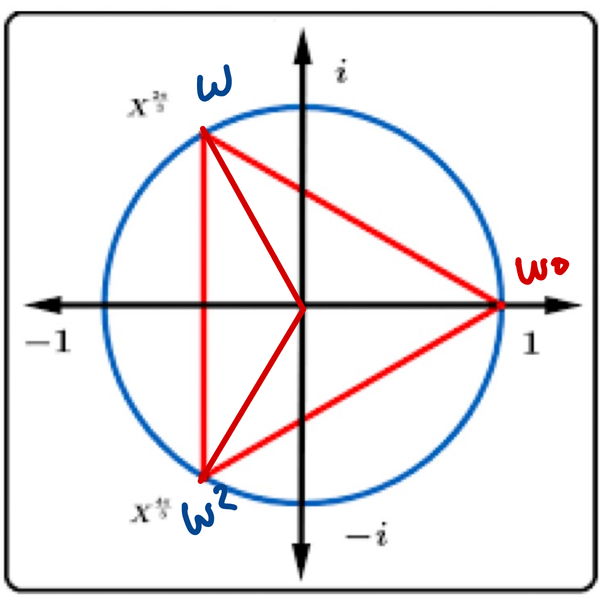
$n=3$ $\quad$ $w_3 = e^{i2n\pi}$

same stop

$w_0 = z = 1$

$w = e^{i2\pi/3}$

$w^2 = e^{i4\pi/3}$



$z^4=1$

$$z^4-1 = (z^2+1)(z^2-1)$$
$$=(z^2-i^2)(z+1)(z-1)$$
$$=(z+i)(z-i)(z+1)(z-1)$$

solution $z=\pm 1$ $\quad z=\pm i$

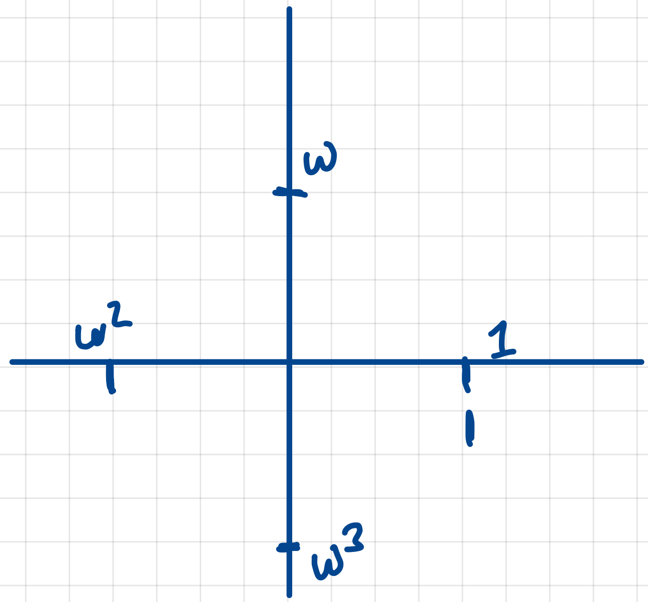
$$z^4=e^{i2n\pi}$$
$$(z^4)^{1/4}=(e^{i2n\pi})^{1/4}$$
$$z=e^{in\pi/2}$$

$n=0 \quad z=1$

$w \quad n=1 \quad z=e^{i\pi/2}$

$w^2 \quad n=2 \quad z=e^{i\pi}$

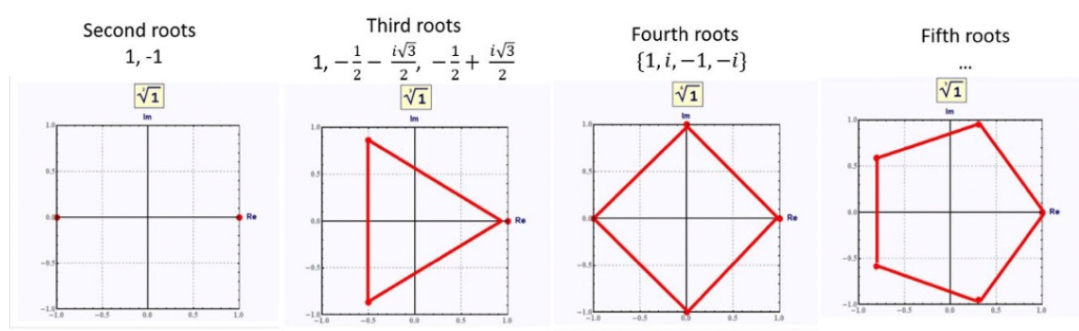
$w^3 \quad n=3 \quad z=e^{i\frac{3\pi}{2}}$



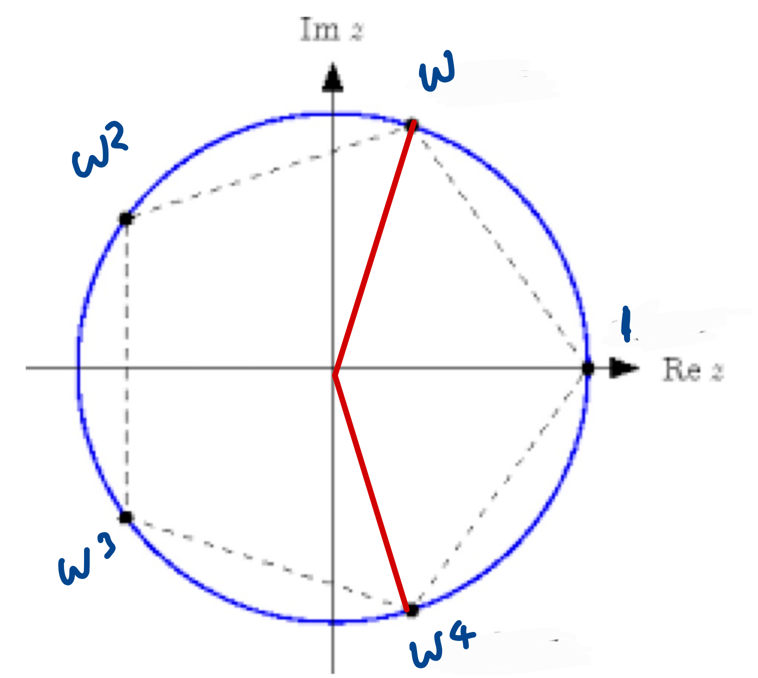
geometric series

$$1+w+w^2+w^3 = \frac{w^4-1}{w-1}$$

$$(w-1)(1+w+w^2+w^3) = w^4-1$$



$z^5 = 1$



$w = e^{i 2\pi/5}$

$w^2 = e^{i 4\pi/5}$

$w^3 = e^{i 6\pi/5}$

$w^4 = e^{i 8\pi/5}$

$w + w^4 =$ a real quantity

find $\sqrt{1+i}$

find the square roots of $1+i$

two way

start with $z=\sqrt{1+i}$

method 1

$z=a+ib$

$a+ib=\sqrt{1+i}$

$(a+ib)^2=1+i$

$a^2-b^2+2iab=1+i$

equating the real and imaginary parts $\Rightarrow$

$a^2-b^2=1$ (A)

$2ab=1$ (B)

$a=\frac{1}{2b}$

$a^2=\frac{1}{4b^2}$

$\frac{1}{4b^2}-b^2=1$

$1-4b^4=4b^2$ let $u=b^2$

$0=4u^2+4u-1$

$u=\frac{-4\pm\sqrt{16-64}}{8}$

$b^2=\frac{-4\pm\sqrt{-48}}{8}$

$b=\sqrt{\frac{-4\pm i\sqrt{48}}{8}}$

method 2

$z^2=1+i$

$z^2=\sqrt{2}\,e^{i(\pi/4+2n\pi)}$

$(z^2)^{1/2}=\left(2^{1/2}e^{i(\pi/4+2n\pi)}\right)^{1/2}$

$z=2^{1/4}e^{i(\pi/8+n\pi)}$

$z=2^{1/4}e^{i\pi/8}$

$z=2^{1/4}e^{i9\pi/8}$

$z=2^{1/4}e^{i\pi/8}$

$=2^{1/4}\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)$

$\cos 2\theta=2\cos^2\theta-1$

$\theta=\pi/8$

$\cos\pi/4=2\cos^2\frac{\pi}{8}-1$